

# Anomaly in the path integral formulation of the Langevin dynamics

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## ABSTRACT

We study a dissipative Langevin dynamics in the path integral formulation using the Martin-Siggia-Rose formalism. The effective action is supersymmetric and we identify the supercharges and derive corresponding Ward identities. We explicitly study the Ornstein-Uhlenbeck process, a Gaussian example of Langevin dynamics and show that two Ward identities are anomalous. Finally by invoking the supersymmetric localisation method we argue that this is a universal phenomenon in the Langevin dynamics.

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# 1 Introduction

Symmetries present at the level of classical actions can be broken by microscopic effects. In quantum physics it leads to a non-conservation of Noether currents. This phenomenon is known as quantum anomaly [1]. In a modern formulation one defines a quantum theory by a path integral - an integral of the exponentiated classical action over quantum trajectories. If a given symmetry is anomalous the measure of integration is not invariant and produces the anomaly upon symmetry transformation [2]. Technically the symmetry is broken by a cut-off introduced in the regularisation procedure. For example in the case of chiral anomalies ultraviolet regularisation breaks the axial current conservation. The chirality injected at high momenta propagates to the larger scales.

Because anomalies are purely quantum effects it was believed that they will not have macroscopic manifestations. However, it turned out not to be the case. A manifestation of a chiral anomaly was suggested in the hydrodynamic regime of a system of massless Weyl fermions [3–6]. In this case the chiral anomaly leads to a transport properties, which are absent for non-anomalous systems. In fact hydrodynamics was the first theory which came across incarnation of anomalies in turbulent flows although such a point of view was provided only once the precise meaning of anomalies was understood in high-energy physics. In turbulence viscosity is an effective cut-off which parameterizes microscopic degrees of freedom. It breaks the time reversal symmetry and the energy conservation. When viscosity goes to zero one would expect that the symmetry is restored. However, in the inviscid limit the velocity is not smooth [7] and time reversal symmetry is not restored even when viscosity and the dissipation scale go to zero. The dissipation rate has a finite limit, which gives an energy flux through scales. This phenomenon is known as dissipative anomaly [8–14]. Although similar in spirit the dissipative anomaly in viscous systems is formulated in a different language than quantum anomalies. It is usually stated at the level of stochastic equations of motion. The most understood example is the Burgers equation in  $1+1$  dimensions. In this case one can explicitly construct solutions and take the inviscid limit. The general question that has not been addressed yet is whether classical anomalies can be formulated in a similar way as quantum anomalies. In other words can we formulate an effective action and the corresponding path integral that has the property that the integration measure breaks a symmetry of the action. To address this question it is required to have a better understanding of the path integral formulation of dissipative stochastic systems. We note that there has been a recent progress in this direction using effective actions [15–23] and the so-called Martin-Siggia-Rose (MSR) formalism [24].

In this work we will study the over-damped Langevin dynamics with noise that belongs to a subclass of potential or gradient systems [25]. Its properties can be used to understand more complicated stochastic evolution. It is well known that the Langevin dynamics can be formulated in terms of path integrals using MSR construction. Therefore we can view the Langevin dynamics as a toy model that illustrates more general features of stochastic systems. In fact Burgers equation is an example of Langevin, that belongs to a more general non-gradient Langevin dynamics. The effective action of over-damped, potential Langevin equation constructed from MSR possesses a number of symmetries [26–28]. It can be linked to the microscopic Schwinger-Keldysh field theory which gives rise to Langevin dynamics in the classical limit [29]. A peculiar feature of this symmetry is that it mixes physical and the ghost fields present in the theory. Therefore it is usually stated that the effective action for Langevin dynamics is supersymmetric.

By an explicit example of the Langevin dynamics, the Ornstein-Uhlenbeck process [30], we will show that this symmetry is not preserved by the path integral, thus leading the anomaly. Finally we will give an argument that the anomaly is present in more general systems by studying supersymmetric localisation [31, 32].

## 2 Particle on a Schwinger-Keldysh contour

The Schwinger-Keldysh formalism was developed to calculate the non-equilibrium correlation functions. In equilibrium in order to calculate correlation functions we use conventional perturbation theory. However, in non-equilibrium we do not have the usual control over the final state. The time-ordered correlation function reads

$$iG(\mathbf{x}, t; \mathbf{x}', t') = \langle \Phi(\infty) |_F T[\mathcal{S}(\infty, -\infty) \phi(\mathbf{x}, t) \phi^\dagger(\mathbf{x}', t')] | \Phi(-\infty) \rangle_I, \quad (2.1)$$

where  $\mathcal{S}(\infty, -\infty)$  is the  $S$ -matrix and the assumption that the final state differs from initial state only by a phase is broken out-of-equilibrium. A method used to avoid dealing with the quantum state at infinity is to evolve back to the initial state

$$| \Phi(-\infty) \rangle_I = S(-\infty, \infty) | \Phi(\infty) \rangle_F \quad (2.2)$$

and introduce a two-branch contour together with a contour ordering  $T_c$ . In fact we do not even need to consider infinite past if we know the density matrix at some finite time  $t_0$ . Then we can evolve our system up to some finite time and then back to  $t_0$ . The evolution contour is now closed in time. Finally we can include finite temperature effects by adding the third imaginary time branch, which will implement the thermal boundary conditions. We can use the contour to define an effective functional that will generate the relevant correlation functions

$$Z[H_1, H_2] = \int [d\phi_1][d\phi_2] \exp[i(S[\phi_1, H_1] - S[\phi_2, H_2])] \exp[iS_{IF}(\phi_1, \phi_2)], \quad (2.3)$$

where  $\phi_i$  are defined on the upper and lower branches of the contour and  $S_{IF}$  is an interaction between different copies of the system. Differentiating with respect to the sources gives a matrix of Green's functions. We see from (2.3) that the Schwinger-Keldysh construction doubles the degrees of freedom. This creates certain redundancies in the description, which can be understood as a gauge symmetry [18]. This means that we can make redefinitions of fields and the physical observables remain unchanged. One possible choice of such a redefinition is the Keldysh rotation

$$\phi_r = \frac{1}{2}(\phi_1 + \phi_2), \quad \phi_a = \phi_1 - \phi_2, \quad (2.4)$$

$$H_r = \frac{1}{2}(H_1 + H_2), \quad H_a = H_1 - H_2. \quad (2.5)$$

$r$ -type operators are conjugate to  $a$ -type sources and vice versa. A consequence of such redefinitions is the existence of symmetry charges that act on functionals. This symmetry enforces constraint on the correlation functions. For example if we align sources then the cyclicity of the trace and the unitarity imply that the partition function is independent of  $H$ . Therefore if  $a$ -type sources are set to zero the variations with respect to  $r$  sources must vanish and the

partition function becomes topological. As a result a Schwinger-Keldysh partition function is invariant under two topological BRST charges  $\mathcal{Q}_{SK}$  and  $\bar{\mathcal{Q}}_{SK}$ , which enforce the above constraint. Apart from this topological symmetry the partition function has an additional symmetry if the initial state is thermal, which corresponds to the time evolution in imaginary time

$$Z[H_1(t_1), H_2(t_2)] = \text{Tr} \left( U_2^\dagger[H_2(t_2)] e^{\beta \mathcal{H}} U_1[H_1(t_1 - i\beta)] \right). \quad (2.6)$$

From (2.6) we can obtain the Kubo-Martin-Schwinger (KMS) condition for the thermal correlators. This symmetry is generated by two charges which we denote  $\mathcal{Q}_{KMS}$  and  $\bar{\mathcal{Q}}_{KMS}$ . It was noted that the four symmetry generators form an algebra, which has been previously encountered in the topological field theory literature and goes by the name of the extended  $\mathcal{N}_T = 2$  equivariant cohomology algebra. Finally the KMS symmetry is combined with CPT invariance, which is spontaneously broken to obtain dissipative effects.

Let us see how we can formulate a particle dynamics on the Schwinger-Keldysh contour. We start with the action

$$S[\varphi] = \int dt \left[ \frac{1}{2} \dot{\varphi}^2 - V(\varphi) \right], \quad (2.7)$$

and split the field into two components  $\phi_1$  and  $\phi_2$  residing on a two-branch contour according to (2.4). In terms of these new fields the action takes the form

$$S[\phi_r, \phi_a] = - \int dt \left[ \phi_a \ddot{\phi}_r - V(\phi_r + 2\phi_a) + V(\phi_r - 2\phi_a) \right], \quad (2.8)$$

where we performed the integration by parts. If we assume that the fluctuations of the  $a$  component are small we can expand the potential terms to get

$$S[\phi_r, \phi_a] = - \int dt \left[ \phi_a \left( \ddot{\phi}_r + \frac{\partial V(\phi_r)}{\partial \phi_r} \right) \right]. \quad (2.9)$$

We notice that we can perform the integration over  $\phi_a$  in the partition function,

$$Z = \mathcal{N} \int [d\phi_r] \delta \left( \ddot{\phi}_r - \frac{\partial V(\phi_r)}{\partial \phi_r} \right) \quad (2.10)$$

which gives the equation of motion for the  $r$  field

$$\ddot{\phi}_r = - \frac{\partial V(\phi_r)}{\partial \phi_r}. \quad (2.11)$$

As we will see later this form of the action resembles the effective action obtained for Langevin dynamics without noise. To obtain the noise contribution one has to carefully take into account quantum fluctuations and take  $\hbar \rightarrow 0$  limit [29]. Therefore we can view the Langevin equation as coming from the Schwinger-Keldysh construction and we expect it to be invariant precisely under the  $\mathcal{N}_T = 2$  symmetry, which we will identify as the Parisi-Sourlas supersymmetry [28].

### 3 Stochastic differential equations, MSR formalism and supersymmetry

In our analysis so far we completely ignored the effects of dissipation and fluctuations. Having in mind the Langevin dynamics we want to include these effects in the effective action. It turns out the the one can do that using the so-called Martin-Siggia-Rose (MSR) prescription. In essence one starts with a stochastic differential equation with noise

$$E(\phi(x)) = \nu(x). \quad (3.1)$$

where  $E(\phi)$  is some differential operator and  $\nu(x)$  is a random variable. One must carefully define what does the whole expression mean which is usually done by means of Itô or Stratanovich calculus in a mathematically consistent way. Assuming this we want to calculate the correlation functions for a stochastic process. An efficient tool is to construct a partition function and differentiate with respect to sources. To do that MSR suggest to use the following identity

$$Z[\nu] = \int [dE] \delta(E(\phi) - \nu) = \int [d\phi] \mathcal{J}(\phi) \delta(E(\phi) - \nu). \quad (3.2)$$

$\mathcal{J}(\phi) = \det \frac{\delta E}{\delta \phi}$  is the Jacobian. In this framework  $\phi$  is not a real function of  $x$  but rather a random variable itself.

We will assume that the noise fulfils

$$\langle \nu(x) \nu(x') \rangle = \frac{2\Gamma}{\beta} \delta(x - x') \quad (3.3)$$

i.e. the white noise, with a Gaussian distribution

$$\int [\mathcal{D}\nu] \nu(x) \nu(x') \exp \left( -\frac{\beta}{4\Gamma} \nu^2 \right) = \frac{2\Gamma}{\beta} \delta(x - x') \quad (3.4)$$

In the next steps we introduce an auxiliary field  $\bar{\phi}$  that will give us the delta function in (3.2) and integrate over the noise. The partition function is

$$Z[H, \bar{H}, L, \bar{L}] = \int [\mathcal{D}c] [\mathcal{D}\bar{c}] [\mathcal{D}\bar{\phi}] [\mathcal{D}\phi] \exp \left( \int d\mathbf{x} \Sigma(\phi, \bar{\phi}, c, \bar{c}) + \bar{H}\phi + H\bar{\phi} + L\bar{c} + c\bar{L} \right), \quad (3.5)$$

where we expressed the Jacobian as an integral over ghost fields and introduced sources for every field. The effective action  $\Sigma$  given by

$$\Sigma(\phi, \bar{\phi}, c, \bar{c}) = -\frac{\Gamma}{\beta} \bar{\phi}^2 - i\bar{\phi}E(\phi) + c\frac{\delta E}{\delta \phi}\bar{c} \quad (3.6)$$

The effective action constructed for stochastic differential equations has three auxiliary fields - one real and two Grassmannian. In addition to that we see that the form of this action resembles equation (2.9) upon identification  $\phi_a \rightarrow \bar{\phi}$ . This strongly suggests that we can interpret stochastic dynamics as emerging from microscopic Schwinger-Keldysh construction. Up to this point

our considerations are completely general. Now we will restrict our attention to the Langevin dynamics. If equation (3.1) has form:

$$E(\phi) = \partial_t \phi + \Gamma \frac{\delta}{\delta \phi} U(\phi) \quad (3.7)$$

the SDE is an over-damped, purely dissipative Langevin equation. It is a valid approximation when the inertia of the particle is negligible in comparison to the linear damping force. One physical realization of this dynamics describes the dynamics of the order parameter of a second order phase transition in axial ferromagnets.

The connection between Langevin dynamics and Schwinger-Keldysh field theories suggest that the effective action possess an underlying  $\mathcal{N}_T = 2$  symmetry structure, which will lead to identities between various correlation functions. This fact was noted a long time ago in the context of dimensional reduction which was later used to unearth various properties and methods to study Langevin dynamics. Explicitly, as shown in [27], the action is invariant under the following transformation:

$$\begin{aligned} Q : \quad & \delta\phi = -\bar{c}\epsilon, \quad \delta c = i\bar{\phi}\epsilon, \quad \text{other variations vanishing} \\ D : \quad & \delta\phi = c\epsilon, \quad \delta\bar{c} = i\bar{\phi}\epsilon, \quad \text{other variations vanishing} \\ \bar{Q} : \quad & \delta\phi = c\epsilon, \quad \delta\bar{c} = (i\bar{\phi} - \frac{\beta}{\Gamma}\dot{\phi})\epsilon, \quad \delta\bar{\phi} = -i\frac{\beta}{\Gamma}\dot{c}\epsilon, \quad \text{other variations vanishing} \\ \bar{D} : \quad & \delta\phi = -\bar{c}\epsilon, \quad \delta c = (i\bar{\phi} - \frac{\beta}{\Gamma}\dot{\phi})\epsilon, \quad \delta\bar{\phi} = i\frac{\beta}{\Gamma}\dot{c}\epsilon, \quad \text{other variations vanishing} \end{aligned} \quad (3.8)$$

In order to make connection with the symmetry algebra of the Schwinger-Keldysh construction for thermal initial states we note that upon identifications

$$\begin{cases} Q = Q_{SK} \\ D = \bar{Q}_{SK} \\ Q - \bar{D} = Q_{KMS} \\ D - \bar{Q} = \bar{Q}_{KMS} \end{cases} \quad (3.9)$$

The above symmetries were recently used to construct effective actions for dissipative hydrodynamics. A natural question arises whether they are true symmetries of the full partition function. We will show, that while  $Q_{SK}$  and  $\bar{Q}_{SK}$  remain symmetries of the path-integrated theory,  $Q_{KMS}$  and  $\bar{Q}_{KMS}$  are in general anomalous. First, we shall compute the partition function for a simple example of Langevin dynamics, the Ornstein-Uhlenbeck process, and show that the Ward-Takahashi (WT) identities associated with  $Q_{KMS}$  and  $\bar{Q}_{KMS}$  do not hold. Then, we will use an argument based on supersymmetric localisation to advocate that the anomaly is indeed a generic phenomenon appearing in this description.

### 3.1 Gaussian theory: Ornstein-Uhlenbeck process

The Ornstein-Uhlenbeck process, describing amongst others thermal noise in RLC circuits, is the simplest of Langevin dynamics and under analytic control. It is defined given as

$$E(\phi(t)) = \dot{\phi}(t) + \Gamma\phi(t) = \nu(t) \quad (3.10)$$

i.e. it is one-dimensional Gaussian model, in a sense that the fields appear in the effective action at most in second powers. We can calculate the partition function by doing the Gaussian

integrals in bosonic and fermionic fields separately

$$Z[\bar{H}, H, \bar{L}, L] = Z_b[\bar{H}, H] Z_f[\bar{L}, L]. \quad (3.11)$$

The result is given by

$$Z_b[\bar{H}, H] = \exp \left( \int \mathbf{d}\tau \mathbf{d}\tau' e^{-\Gamma|\tau-\tau'|} \frac{1}{2\beta} \bar{H}(\tau) \bar{H}(\tau') - i\theta(\tau - \tau') e^{-\Gamma(\tau-\tau')} \bar{H}(\tau) H(\tau') \right), \quad (3.12)$$

$$Z_f[\bar{L}, L] = \exp \left( - \int \mathbf{d}\tau \mathbf{d}\tau' L(\tau) \theta(\tau - \tau') e^{-\Gamma(\tau-\tau')} \bar{L}(\tau') \right). \quad (3.13)$$

with  $\theta$  being a Heaviside step function and  $i$  – imaginary unit.

Traditionally, Ward-Takahashi identities are identities between correlators of fields, derived from symmetry along with assumption that the path integral measure transforms with unit Jacobian, i.e. there is no quantum anomaly. They can be, however stated in terms of identities between variational derivatives of the partition function, see for example [27]. We are going to use those forms in our calculations. The WT identities for symmetries (3.8) read:

$$\begin{aligned} G_{Q_{SK}} &= \int \mathbf{d}\mathbf{x} \bar{H} \frac{\delta}{\delta L} Z + i\bar{L} \frac{\delta}{\delta H} Z = 0 \\ G_{\bar{Q}_{SK}} &= - \int \mathbf{d}\mathbf{x} -\bar{H} \frac{\delta}{\delta \bar{L}} Z + i(\frac{\delta}{\delta H} Z) L = 0 \\ G_{\bar{Q}_{KMS}} &= \int \mathbf{d}\mathbf{x} i\frac{\beta}{\Gamma} H \partial_t \frac{\delta}{\delta \bar{L}} Z - L \frac{\beta}{\Gamma} \partial_t \frac{\delta}{\delta H} Z = 0 \\ G_{Q_{KMS}} &= \int \mathbf{d}\mathbf{x} i\frac{\beta}{\Gamma} H \partial_t \frac{\delta}{\delta L} Z + \frac{\beta}{\Gamma} \bar{L} \partial_t \frac{\delta}{\delta H} Z = 0 \end{aligned} \quad (3.14)$$

Upon using the partition function (3.13) one can evaluate these directly. Then one discovers:

$$G_{Q_{SK}} = \int \mathbf{d}t \mathbf{d}t' \left[ -\bar{H}(t) \theta(t-t') e^{-\Gamma(t-t')} \bar{L}(t') + \bar{L}(t) \theta(t'-t) e^{-\Gamma(t'-t)} \bar{H}(t') \right] = 0 \quad (3.15)$$

$$G_{\bar{Q}_{SK}} = - \int \mathbf{d}t \mathbf{d}t' \left[ -\bar{H}(t) \theta(t'-t) e^{-\Gamma(t'-t)} L(t') + L(t) \theta(t'-t) e^{-\Gamma(t'-t)} \bar{H}(t') \right] = 0 \quad (3.16)$$

$$G_{\bar{Q}_{KMS}} = -\frac{\beta}{\Gamma} \int \mathbf{d}t \mathbf{d}t' L(t) \frac{d}{dt} \left( \frac{1}{\beta} e^{-\Gamma|t-t'|} \bar{H}(t') \right) \neq 0 \quad (3.17)$$

$$G_{Q_{KMS}} = \frac{\beta}{\Gamma} \int \mathbf{d}t \mathbf{d}t' \bar{L}(t) \frac{d}{dt} \left( \frac{1}{\beta} e^{-\Gamma|t-t'|} \bar{H}(t') \right) \neq 0 \quad (3.18)$$

From the above it follows, that WT identities for  $\bar{Q}_{KMS}$  and  $Q_{KMS}$  are not satisfied. Even a Gaussian theory the path integral measure after the transformations generated by those charges is not invariant, so the theory exhibits an anomaly. A particle physicist's intuition suggests here a perturbative argument why should we expect that anomaly to occur universally in stochastic theories: the simple quadratic action of (3.10), in perturbative calculations would serve as a basis to introduce propagators for fields. Then any correlation function would be expressed as a formal (asymptotic) series in some (small) parameter governing deformation of Gaussian theory used to obtain the theory of interest. There should exist a Feynman diagram *at the tree level*

(0th order in perturbation) that breaks the KMS symmetry, and that diagram should be present irrespectively of the nature of perturbation. However, the universality of such an argument is questionable and one usually aims at arguments that do not rely on the detail of a particular theory (a form of the potential term in our case). For that reason let us invoke another argument in favour of universality of anomaly based on the technique called supersymmetric localisation.

## 4 Supersymmetric localisation and the universality of anomaly

In this section we give a brief introduction to the supersymmetric localisation method. The idea behind it resembles saddle-point approximation of integrals provided that the partition function is invariant under a supersymmetry. Since the dissipative Langevin dynamics possess four supercharges let us assume that we have theory defined by a path-integral over both bosonic( $\phi$ ) and fermionic( $\psi$ ) fields:

$$Z = \int [\mathcal{D}\phi] [\mathcal{D}\psi] \exp(S[\phi, \psi]) \quad (4.1)$$

where  $S$  is supersymmetric action and path integral measure transforms under change of variables given by  $\delta_s$  with a unit Jacobian. Also, let  $\delta_s$  be a fermionic symmetry transformation and  $\delta_s^2 = B$  – a bosonic transformation and  $V[\phi, \psi]$  a functional of fields, such that  $BV = \delta_s^2 V = 0$  and bosonic part of  $\delta V$  is positive. Then the supersymmetric localisation principle states that:

$$\frac{d}{d\mu} \int [\mathcal{D}\phi] [\mathcal{D}\psi] \exp(S[\phi, \psi] + \mu \delta_s V[\phi, \psi]) = 0 \quad (4.2)$$

Which implies, that if we let  $\mu \rightarrow \infty$   $Z$  reduces to an integral over critical points of  $\delta_s V$  and a small fluctuation (of WKB type) around them. The set of critical points of  $\delta_s V$  is called *localisation locus* and the part coming from small fluctuations – *1-loop determinant*. For many supersymmetric theories, especially on compact manifolds, the localisation principle allows us to reduce path-integral to a finite dimensional integral. For practical reasons one usually uses  $V$  in form

$$V = \sum_i \psi_i \cdot (\delta_s \psi_i), \quad (4.3)$$

where sum runs over all the fermions in theory and  $\cdot$  denotes some scalar product (depending on theory, it may require for example using Hermitian conjugation). Then

$$\delta_s V|_{\text{bos}} = \sum_i (\delta_s \psi_i) \cdot (\delta_s \psi_i) \quad (4.4)$$

which is manifestly positive and quadratic so that the locus is given by

$$(\delta_s \psi_i)|_{\psi=\psi_0, \phi=\phi_0} = 0. \quad (4.5)$$

Here, we denote locus field configuration as  $\psi_0, \phi_0$ . These fields will dominate the path integral in the limit  $\mu \rightarrow \infty$ . To compute potential corrections, co-called 1-loop corrections, we redefine fields

$$\psi_i = \psi_{0,i} + \mu^{-1/2} \tilde{\psi}_i \quad (4.6)$$



and then what survives the large  $t$  limit is

$$Z = \int_{\text{loc}} [\mathcal{D}\phi_0] [\mathcal{D}\psi_0] \exp(S[\phi_0, \psi_0]) \int [\mathcal{D}\tilde{\phi}] [\mathcal{D}\tilde{\psi}] \exp\left(\frac{\delta^2}{\delta\phi\delta\phi} \delta_s V|_{\phi_0}\right) = \int_{\text{loc}} [\mathcal{D}\phi_0] [\mathcal{D}\psi_0] \exp(S[\phi_0, \psi_0]) \text{SDet}(\delta_s V(\phi_0))^{-2}, \quad (4.7)$$

where the last line is just changing notation to more compact one. Let us also note, that this method allows us to compute correlation functions of supersymmetric operators, i.e ones for which

$$\delta_s \mathcal{O} = 0, \quad (4.8)$$

where  $\mathcal{O}$  is the operator. This can be easily seen by observing that deforming action in (4.1) by a source term for such an operator

$$S[\phi, \psi] \rightarrow \tilde{S}[\phi, \psi, J] = S[\phi, \psi] + \int \mathbf{d}x J \mathcal{O} \quad (4.9)$$

does not affect any of the assumptions mentioned before (4.2).

## 4.1 Computation for the Langevin dynamics

We have shown that a particular example of the Ornstein-Uhlenbeck process exhibits an anomaly of the KMS supersymmetry. Now we will argue that this property should be universal in the gradient Langevin dynamics. To that we will employ the supersymmetric localisation introduced above, a non-perturbative tool that originates from field theories. The dissipative Langevin dynamics is invariant supercharges at the level of the action. If the measure is invariant under these supercharges the calculation of the correlation functions will be simplified. However, based on the example of the Ornstein-Uhlenbeck process we expect the measure actually breaks the KMS supersymmetry and the path integral will localise on the wrong solutions. We will see that this is indeed the case and the KMS solution on which the path integral localises is time-independent which contradicts numerical calculations [33].

To make the above statements precise, we need to find a supersymmetry transformation that allows us a computation of some interesting correlation functions. Let us take a combination:

$$\delta_s = \frac{\Gamma}{\beta}(D - \bar{Q}) = \frac{\Gamma}{\beta}(\bar{Q}_{KMS}) : \begin{cases} \delta\phi = 0 \\ \delta\bar{c} = \dot{\phi}\epsilon \\ \delta\bar{\phi} = i\dot{\bar{c}}\epsilon \\ \delta c = 0 \end{cases} . \quad (4.10)$$

Then  $\delta_s$  is nilpotent ( $\delta_s^2 = 0$ ) and, what is more important any (reasonable) function of field  $\phi$  (which is our physical field) is supersymmetric

$$\delta_s F(\phi) = F'(\phi) \delta_s \phi = 0. \quad (4.11)$$

Taking standard choice of  $V$  (4.3) with  $\psi_i = \{c, \bar{c}\}$ , we can compute the locus

$$\dot{\phi}_0 = 0. \quad (4.12)$$

That result means, that the path integral (4.1) for  $Z[0, \bar{H}, 0, 0]$  is obtained by integrating over *time independent* field configurations. Which would also mean that

$$\forall_i \frac{d}{dt_i} \langle G(\phi(t), \phi(t_2) \dots \phi(t_n)) \rangle = 0, \quad (4.13)$$

for any function  $G$  of field  $\phi$ . The above calculation shows that if KMS supersymmetry were a true both of the action and the measure correlation functions would localise on the time-independent configurations. We know that this cannot be the case as one can check using numerical simulations [33]. Since it was explicitly shown that KMS is a symmetry of the action we conclude that for a general potential Langevin dynamics the path integral measure breaks the KMS supersymmetry. As a result the time dependence of the correlation functions is universally determined by this stochastic KMS anomaly.

## 5 Summary and outlook

In the construction of effective theories the crucial question is what symmetries are present at a given scale. Analysing Schwinger-Keldysh formulation of field theories it seems natural to expect that the effective description should have gauge invariance that follows from the doubling of degrees of freedom as well as the symmetry that will give KMS condition at the level of the correlation function for thermal states. These symmetries are realised as supersymmetries which is easily visible in the superspace. An example in which these symmetries are realised at the level of effective actions in a straightforward manner is the Langevin dynamics. In this Letter we have explicitly shown that supersymmetry associated with the KMS condition is broken by the path integral measure, thus leading to an anomaly. We have shown these at the level of WT identities in the simplest, quadratic Ohnstein-Uhlenbeck model. Then using argument based on the method of supersymmetric localisation, we argued that the presence of an anomaly is an universal phenomenon for a broad class of purely dissipative Langevin form equations. Our result provides the first construction of an anomaly in a dissipative system based on the path integral formulation. Since the Langevin dynamics can be obtained as an approximate description in the long-wavelength regime it is natural to expect that a similar anomaly should be present in fluids. To gain further insight into the form of anomalies it might be necessary to understand the Fujikawa procedure in the context of dissipative systems. Perhaps this can be achieved by a generalisation of methods developed in the context of supersymmetric field theories [34]. The anomaly should emerge after a proper regularisation of the measure with respect to the time translation symmetry. Since the anomaly captures the time evolution of dissipative systems it is perhaps possible to reformulate the dissipative anomaly in the path integral language.

Chiral anomalies in particle physics can be understood as manifestation of a regularisation failure. In order to keep a gauge symmetry at a quantum level we need to sacrifice a chiral symmetry. There exist no regularization prescription that will preserve both symmetries. As was shown recently in [18] the Schwinger-Keldysh symmetry generated by  $Q_{SK}$  and  $\bar{Q}_{SK}$  is in fact a gauge symmetry. Perhaps in the stochastic systems Kubo-Martin-Schwinger symmetry plays a similar role as a chiral symmetry in high-energy physics and one cannot regularize the system to preserve both symmetries. Another direction, which we think is promising, concerns

the status of the KMS anomaly in the full Schwinger-Keldysh theory and perhaps generalisations to out-of-time contours in models that describe chaotic systems. This would indicate that the presence of anomalies may play some role in obtaining non-unitary effective large scale theories from microscopic, unitary theories.

## Appendix A: Superspace formulation

Most of calculations, especially concerning the WT identities were done in superspace formalism, which we avoided in the main text for brevity. However, we used conventions that are slightly unorthodox, therefore we quote them here. Our superfield is

$$\Phi = \phi + \theta\bar{c} + c\bar{\theta} + \theta\bar{\theta}i\bar{\phi}. \quad (5.1)$$

In order to have the proper source term namely

$$\int \mathbf{d}\mathbf{x} d\bar{\theta} d\theta J\Phi = \int \mathbf{d}\mathbf{x} \bar{H}\phi + H\bar{\phi} + L\bar{c} + c\bar{L}, \quad (5.2)$$

our super source  $J$  must be

$$J = -iH + \bar{L}\theta + \bar{\theta}L + \theta\bar{\theta}\bar{H}. \quad (5.3)$$

The symmetry generators are listed in the table below.

$$\begin{aligned} D &= \partial_{\bar{\theta}} & \bar{D} &= \partial_{\theta} - \frac{\beta}{\Gamma}\bar{\theta}\frac{d}{dt} \\ Q &= \partial_{\theta} & \bar{Q} &= \partial_{\bar{\theta}} + \frac{\beta}{\Gamma}\theta\frac{d}{dt} \\ Q_{KMS} &= \frac{\beta}{\Gamma}\bar{\theta}\frac{d}{dt} & \bar{Q}_{KMS} &= -\frac{\beta}{\Gamma}\theta\frac{d}{dt} \end{aligned} \quad (5.4)$$

Now partition function (3.5) for a system of our interest can be written as:

$$Z[H, \bar{H}, L, \bar{L}] = \int [\mathcal{D}\Phi] \exp \left( \int \mathbf{d}\mathbf{x} d\bar{\theta} d\theta - \frac{1}{\beta} \bar{D}\Phi D\Phi - U(\Phi) + J\Phi \right), \quad (5.5)$$

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